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The non-relativistic Lorentz and Coulomb gauges in the region of a mirror

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Abstract. The non-relativistic theory of the Lorentz gauge is developed for the region of a perfectly conducting surface using the Gupta-Bleuler indefinite metric and a non-orthogonal set of basis vectors. The Lorentz gauge is transformed into the Coulomb gauge by unitary transformation and in so doing expressions are obtained for the Coulomb energy and its special case the image force. The two non-physical photons of the Lorentz gauge are found to exist also in the Coulomb gauge.

1. Introduction

The Coulomb gauge is very different to the Lorentz gauge. It is not at first sight Lorentz invariant, it is a non-local gauge involving the strange Coulomb force acting instantaneously across space and it normally has but two quantised photon fields instead of the four of the Lorentz gauge. Nevertheless it is possible to transform the Lorentz gauge into the Coulomb gauge by a unitary transformation and thus to demonstrate the equivalence of the underlying theories. In the present paper this transformation is performed for a half-space bounded on one side by a plane perfectly conducting surface. In so doing, an expression for the classical image force is obtained purely from quantum mechanical considerations, thus dispelling the ambiguity with which it has previously been surrounded.

The Lorentz gauge has not often been employed in the theory of conducting surfaces because of its extra complexity; however, Babiker (1982) has introduced this gauge to the field in a non-relativistic form that he hoped might be useful in cases where the Coulomb gauge is problematical due to difficulties over the Coulomb terms in the Hamiltonian. Babiker used the Lorentz gauge to obtain an expression for the image force but by the use of a perturbation method only.

The modern theory of the Lorentz gauge is the Gupta-Bleuler theory (Gupta 1950, Bleuler 1950). The state space in this theory is defined on an indefinite metric, the Gupta-Bleuler metric (a good introduction to the theory of metrics is given by Pandit (1959)). The Gupta-Bleuler metric is not to be confused with the Minkowski metric of relativity—although the two metrics are not independent. Use of the indefinite metric allows the four components of the vector potential to be quantised symmetrically,

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thus there are four independent photon fields in this theory, two that correspond to the classical transverse modes and two additional virtual fields without direct physical existence. All Coulomb interactions in the Lorentz gauge occur through the medium of these virtual fields, which act as exchange particles. The Coulomb gauge differs from the Lorentz gauge in that only the two transverse modes are normally quantised. The time component of the vector potential, the electrostatic potential, remains unquantised, governed as in the classical theory by the Poisson equation. Coulomb interactions arise by quite different mechanisms in the two gauges. There are normally no virtual fields in the Coulomb gauge and Coulomb interactions arise instead through the electrostatic potential via a special term in the Hamiltonian, the Coulomb term. The Coulomb term must be determined for each new situation through a solution of the Poisson equation, and it is this that causes trouble with conducting surfaces, particularly of electronically dense materials.

The equivalent of the classical gauge transformation in the quantum theory takes the form of a unitary transformation. Babiker and Loudon (1983) discuss this equivalence, performing a particular transformation by both the classical and quantum mechanical methods and comparing the results. They also show that a further method, that of adding a total time derivative to the Lagrangian, is the equivalent of the other two.

The transformation from the Lorentz to the Coulomb gauge has been carried out by various authors employing various methods, although prior to the present work it does not seem to have been performed by unitary transformation in a non-relativistic form or to have been performed at all for the presence of a conducting surface.

Babiker and Loudon perform the transformation formulated as a gauge change in the classical theory. Heitler (1954) also performs the transformation using the classical theory, but formulated in the language of a canonical transformation. Sometimes the transformation is referred to as the elimination of the longitudinal field; for instance by Schwinger (1948) using a quantum mechanical method but writing prior to Gupta's formulation of the Lorentz gauge. The unitary transformation has been performed in relativistic formulation by Bleuler (1950) in the paper which contributed to the Gupta-Bleuler method, and more recently by Durr and Rudolph (1969) using the ghost-state formulation employed in the present paper. The transformation from the Lorentz to the Coulomb gauge is a transformation from a local to a non-local theory; this situation is discussed by Ascoli and Minardi (1958).

The first part of the present work is occupied in developing the formalism of the Lorentz gauge as it pertains to a half-space. A non-relativistic theory is employed for the motion of massive particles, although the electromagnetic field is treated relativistically with the full second-quantised approach. A non-orthogonal set of basis vectors is used in the description of the electromagnetic field, employing two zero-norm ghost states. A more complete account of the present work, and in particular of the Gupta theory, is given by Hart (1985).

Relativistic notation is employed throughout, such that the arbitrary 4-vector f^μ is expressed in the form $f^\mu = (f^0, \boldsymbol{f}) = (f^0, \boldsymbol{f}_\parallel, f_\perp)$, where $\boldsymbol{f}_\parallel$ and $f_\perp$ are the components of f^μ parallel and perpendicular, respectively, to the surface of the mirror and $\boldsymbol{f}_\parallel$ is a two-dimensional vector. The vector $\hat{f}^\mu$ is also used, where $\hat{f}^\mu = (f^0, \boldsymbol{f}_\parallel, -f_\perp)$.

The summation convention will be assumed, unless otherwise stated, such that repeated roman indices take the values 1, 2, 3 and repeated greek indices the values 0, 1, 2, 3. A perfectly conducting mirror surface is assumed to pass through the origin, with axes oriented such that the perpendicular to the surface of the mirror is parallel to the axis carrying the label 3 with the positive section in the empty half-space, so that $f_\perp = f^3$.

Contravariant and covariant quantities are distinguished by raising and lowering the indices, and are related through the metric tensor, $g^{\mu\nu}$ $(-+++)$, such that $f^\mu = g^{\mu\nu} f_\nu$. The contravariant set is the set directly comparable to the set of non-relativistic quantities; for example, time is given by $t = x^0$ and the derivative $\partial^0 f = \partial f/\partial x_0 = -\partial f/\partial t$.

Whenever there could be confusion an index without Lorentz significance is placed in brackets, thus $T^{(\mu)}$. These indices are merely labels and their location in the raised or lowered position is without significance. The summation convention does not apply to indices in brackets.

The Dirac delta function and the Kronecker delta take the notations $\delta(f)$ and δ_{ij}, respectively.

Heaviside units are employed throughout with the speed of light and Planck's constant equal to unity.

2. The Lorentz gauge in the half-space

The usual non-relativistic Hamiltonian can be employed without change in the half-space:

$$H = H_{\text{field}} + H_1 + H_\phi \tag{1a}$$

where

$$H_{\text{field}} = \sum_\nu \int_{\frac{1}{2}} \mathrm{d}^3\boldsymbol{k}(N^{(\nu)}(k) + \tfrac{1}{2}))\omega \tag{1b}$$

$$H_1 = (\boldsymbol{p} - \boldsymbol{A}(r))^2/2m \tag{1c}$$

$$H_\phi = qA^0(r) \tag{1d}$$

with A^μ the 4-vector potential, $N^{(\nu)}(k)$ the number operator corresponding to mode ν of the photon field, and r^μ and $\boldsymbol{p} = -\mathrm{i}\boldsymbol{\nabla}_r$ the position vector and momentum operator, respectively, of a particle of charge q and mass m in interaction with the field. The subscript of $\frac{1}{2}$ on the integral sign denotes a definite integral over a half-space on the positive axis. The wavevector k^μ fulfils an identical role to the corresponding wavevector in the full-space, its time component having the same free-field value, $\omega = |\boldsymbol{k}|$, its perpendicular component, however, not taking negative values, $k_\perp \geq 0$.

The substance of the conductor is assumed to perfectly exclude all electric and magnetic fields, leading to boundary conditions on its surface in the form

$$\partial^1 A^0 = \partial^0 A^1 \qquad \partial^2 A^0 = \partial^0 A^2 \qquad \partial^1 A^2 = \partial^2 A^1. \tag{2}$$

These conditions do not necessarily require the vector potential also to vanish inside the mirror, and, strictly speaking, it probably ought not to. It is, however, a worthwhile simplification for the present purposes to assume that it does.

It is convenient to define a function $T^{(\mu)}$ such that

$$T^{(0)}(u) = T^{(1)}(u) = T^{(2)}(u) = \mathrm{i}\sin(u) \qquad T^{(3)}(u) = \cos(u) \tag{3}$$

and a step function $\theta(x_\perp)$ which takes unit value for $x_\perp \geq 0$ but vanishes elsewhere. Then a convenient decomposition of the vector potential that satisfies the boundary conditions (see Barton (1974) for a similar decomposition in the Coulomb gauge) becomes

$$A^{\mu}(x)=\theta(x_{\perp})\int_{\frac{1}{2}}\mathrm{d}^{3}\boldsymbol{k}\,M(k)a^{\mu}(k)\exp(\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{x}_{\parallel}-\mathrm{i}\omega t)T^{(\mu)}(k_{\perp}x_{\perp})+\text{adjoint} \tag{4}$$

where $a^{\mu}(k)$ and $a^{\mu\dagger}(k)$ are the annihilation operators of the photon field and their adjoints, the adjoint being with respect to the Gupta–Bleuler metric rather than the usual Hermitian conjugate. The normalisation factor $M(k)$ is conveniently chosen such that $M(k)=(4\pi^{3}\omega)^{-1/2}$.

The photon operators $a^{\mu}(k)$ and $a^{\mu\dagger}(k)$ are both 4-vectors, and this must be taken into account when it is necessary to ensure relativistic invariance. In particular, the commutation relations of these operators become

$$[a^{\mu}(k),a^{\nu}(k')]=0=[a^{\mu\dagger}(k),a^{\nu\dagger}(k')] \tag{5a}$$

$$[a^{\mu}(k),a^{\nu\dagger}(k')]=g^{\mu\nu}\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{5b}$$

and the photon number operators

$$N^{(\nu)}(k)=a^{\nu\dagger}(k)a_{\nu}(k)=\xi^{(\nu)}a^{\nu\dagger}(k)a^{\nu}(k)\qquad\text{not summed over }\nu \tag{6}$$

where $\xi^{(\nu)}=\pm1$ is a parameter taking a negative value for the time-like mode, $\xi^{(0)}=-1$, and being otherwise positive.

The occurrence of the factors $g^{\mu\nu}$ and $\xi^{(\nu)}$ is ultimately the reason behind the use of the indefinite metric, for it is easily seen that equations (5) and (6) lead to a state space in which the time-like photon can have a negative norm:

$$(n^{(\nu)}|n^{(\nu)})=(\xi^{(\nu)})^{n^{(\nu)}} \tag{7}$$

where the state $|n^{(\nu)})$ is an eigenstate of $N^{(\nu)}$ with eigenvalue $n^{(\nu)}$, such that

$$|n^{(\nu)})=(n^{(\nu)}!)^{-1/2}(\xi^{(\nu)}a^{\nu\dagger})^{n^{(\nu)}}|0) \tag{8}$$

with the state $|0)$ representing the photon vacuum. The round bracket symbols are employed instead of the more usual Dirac bras and kets in order to emphasise the use of an indefinite metric.

With an indefinite metric, expectation values of the number operators, and therefore of the Hamiltonian, may be negative even though their eigenvalues are positive, thus raising the possibility of negative energies. Gupta (1950), however, was able to use the Lorentz condition in a special form, the Gupta condition, to prevent the occurrence of negative energy states in the physical domain. The Gupta condition as derived for a full-space, however, is not applicable to the half-space and must be derived anew from the Lorentz condition.

The classical Lorentz condition, $\partial^{\mu}A_{\mu}=0$, cannot be applied to the quantum operators directly because to do so would mean equating operators that obey different commutation rules. The alternative is to apply the condition only to the expectation values:

$$\partial^{\mu}(\text{physical}|A_{\mu}(x)|\text{physical})=0 \tag{9}$$

where $|\text{physical})$ is any physically observable state. Then, because the vector potential is self-adjoint, equation (9) is equivalent to

$$\chi^{-}(x)|\text{physical})=0 \tag{10}$$

where $\chi^{-}=\partial^{\mu}A_{\mu}^{-}$ with A_{μ}^{-} the part of A_{μ} containing only annihilation operators.

Equation (10) is the celebrated Gupta condition. It is not a condition on the operators but rather on the state vectors. This creates a situation quite different from the classical one. In the classical theory the imposition of the Lorentz condition, by setting an initial condition, removes a degree of freedom from the fields; no freedom is removed in the Gupta theory, however, but instead the set of state vectors is divided into two subsets—those that obey the condition, and those that do not. It is basic to the Gupta theory that only the former represent states directly detectable by physical measurement; states that do not obey the condition are to be considered virtual, with no direct physical effect.

In the presence of electric charge the 4-divergence $\partial^\mu A_\mu^-$ cannot be evaluated directly and it is necessary to use the relationship $\mathrm{d}/\mathrm{d}t\, A^{0-} = \mathrm{i}[H, A^{0-}]$, with H and A^0 from equations (1) and (4). Employing the commutation relations, equations (5), then leads to

$$[k^\mu a_\mu(k) - \mathrm{i}q\theta(r_\perp)M(k)\exp(-\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_\parallel)\sin(k_\perp r_\perp)]|\text{physical}\rangle = 0 \tag{11}$$

where a term involving a delta function at the mirror surface has been ignored (this term can be removed by postulating a certain non-zero vector potential inside the mirror). Equation (11) is the Gupta condition for the half-space; it is comparable with the result for the full-space first obtained by Bleuler (1950).

In the absence of electric charge the Gupta condition becomes more simply

$$k^\mu a_\mu(k)|\text{physical}\rangle = 0 \tag{12}$$

which is identical to the corresponding full-space result.

2.1. *The choice of basis*

The basis can be generalised by writing

$$a^\mu(k) = \sum_\lambda \varepsilon^{\mu(\lambda)}(k)a^{(\lambda)}(k) \tag{13}$$

where the $\varepsilon^{\mu(\lambda)}(k)$ are a set of polarisation vectors and λ is a mode label.

The traditional choice of basis insists on orthogonality, expressed by

$$\varepsilon^{\mu(\lambda)}(k)\varepsilon_\mu^{(\lambda')}(k) = g^{(\lambda)(\lambda')} \tag{14}$$

then sets $k^\mu\varepsilon_\mu^{(1)}(k) = k^\mu\varepsilon_\mu^{(2)}(k) = 0$. This basis is then a generalisation of the classical transverse modes, $\lambda = 1$ and 2, the longitudinal mode, $\lambda = 3$, and the scalar mode, $\lambda = 0$. The transverse modes satisfy the free-field Gupta condition, equation (12), as operators, even though the condition is normally stated to be a condition on the state vectors only. Thus equation (12) can be written:

$$[a^{(3)}(k) - a^{(0)}(k)]|\text{physical}\rangle = 0. \tag{15}$$

This implies that in the absence of electric charge both longitudinal and scalar modes can appear in physical states, but only in pairs, one of each.

Equation (15) suggests a possible replacement of the longitudinal and scalar modes by two new modes, defined such that

$$a^{\mathrm{g}}(k) = 2^{-1/2}[a^{(3)}(k) + a^{(0)}(k)] \tag{16a}$$

$$a^{\mathrm{b}}(k) = 2^{-1/2}[a^{(3)}(k) - a^{(0)}(k)] \tag{16b}$$

as first done by Durr and Rudolph (1969). The basis is then composed of the two transverse modes, as before, plus the good ghost, labelled g, and the bad ghost, labelled b. It is then possible to write

$$A^\mu(x) = A^{\mu\mathrm{Tr}}(x) + A^{\mu\mathrm{g}}(x) + A^{\mu\mathrm{b}}(x) \tag{17}$$

where $A^{\mu\mathrm{Tr}}$ is the part of A^{μ} containing the transverse mode operators, $A^{\mu\mathrm{g}}$ contains a^{g} and $a^{\mathrm{g}\dagger}$, and $A^{\mu\mathrm{b}}$ likewise contains a^{b} and $a^{\mathrm{b}\dagger}$; such that, for instance,

$$A^{\mu\mathrm{b}}(x)=\theta(x_{\perp})\int_{\frac{1}{2}}\mathrm{d}^3\boldsymbol{k}\, M(k)\varepsilon^{\mu\mathrm{b}}(k)a^{\mathrm{b}}(k)\exp(\mathrm{i}\boldsymbol{k}_{\|}\cdot\boldsymbol{x}_{\|}-\mathrm{i}\omega t)T^{(\mu)}(k_{\perp}x_{\perp})+\text{adjoint}. \quad (18)$$

It is then useful to define a symbol τ designating any one of the basis, transverse or good or bad ghosts; and to define the creation operators $Cr^{(\tau)}$ such that for the transverse states

$$Cr^{(1)}(k)=a^{(1)\dagger}(k) \qquad Cr^{(2)}(k)=a^{(2)\dagger}(k) \quad (19a)$$

but for the ghost states the labels are reversed

$$Cr^{\mathrm{g}}(k)=a^{\mathrm{b}\dagger}(k) \qquad Cr^{\mathrm{b}}(k)=a^{\mathrm{g}\dagger}(k). \quad (19b)$$

The commutation rules then become

$$[a^{(\tau)}(k),\, a^{(\tau')}(k')]=0=[Cr^{(\tau)}(k),\, Cr^{(\tau')}(k')] \quad (20a)$$

$$[a^{(\tau)}(k),\, Cr^{(\tau')}(k')]=\delta_{\tau\tau'}\delta(\boldsymbol{k}-\boldsymbol{k}') \quad (20b)$$

and the number operators

$$N^{(\tau)}(k)=Cr^{(\tau)}(k)a^{(\tau)}(k) \qquad \text{not summed over } \tau. \quad (21)$$

If now $Cr^{(\tau)}$ is interpreted as the creation operator corresponding to $a^{(\tau)}$ then relationships (20) and (21) take the standard boson form without the factors $g^{\mu\nu}$ or $\xi^{(\nu)}$ interfering.

Such are the intricacies of an indefinite metric that the creation and annihilation operators of the ghost states are not mutually adjoint. The creation operator of the good-ghost state is the adjoint of the bad-ghost annihilation operator, and vice versa, equation (19*b*). Thus

$$|\mathrm{g})=Cr^{\mathrm{g}}|0)=a^{\mathrm{b}\dagger}|0) \qquad |\mathrm{b})=Cr^{\mathrm{b}}|0)=a^{\mathrm{g}\dagger}|0) \quad (22a)$$

with adjoints

$$(\mathrm{g}|=(0|Cr^{\mathrm{g}\dagger}=(0|a^{\mathrm{b}} \qquad (\mathrm{b}|=(0|Cr^{\mathrm{b}\dagger}=(0|a^{\mathrm{g}}. \quad (22b)$$

The confusing nomenclature is clarified by consideration of the annihilation function of the operators; thus when acting to the left $a^{\mathrm{b}\dagger}$ annihilates a bad ghost, $(b|a^{\mathrm{b}\dagger}=(0|$, although a good ghost is created when it acts to the right.

Using the commutation relations it is found that

$$(\mathrm{g}|\mathrm{g})=0=(\mathrm{b}|\mathrm{b}) \quad (23a)$$

$$(\mathrm{b}|\mathrm{g})=1=(\mathrm{g}|\mathrm{b}) \quad (23b)$$

showing that the ghost states are of zero norm and not mutually orthogonal.

It will be of use later to note that just as with an orthogonal basis, for an arbitrary state vector $|m)$ there is one and only one vector $|\hat{m})$ such that $(\hat{m}|m)=1$. However, unlike the orthogonal case, $|\hat{m})$ is not equal to $|m)$ but has the ghost-state occupation numbers reversed:

$$\hat{n}^{\mathrm{Tr}}=n^{\mathrm{Tr}} \qquad \hat{n}^{\mathrm{g}}=n^{\mathrm{b}} \qquad \hat{n}^{\mathrm{b}}=n^{\mathrm{g}} \quad (24)$$

where the n^{τ} and $\hat{n}^{\tau}$ refer to the occupation numbers of the states $|m)$ and $|\hat{m})$, respectively.

2.2. *The physical state space*

Written in terms of the ghost-state operators the Gupta condition, equation (11), becomes

$$[a^b(k) - iq\theta(r_\perp)(M(k)/\omega\sqrt{2})\exp(-i\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_\parallel)\sin(k_\perp r_\perp)]|\text{physical}) = 0 \tag{25}$$

in which the bad-ghost annihilation operator is the only photon operator to appear. Thus the bad ghost is prohibited absolutely from entering physical states except in certain close associations with electric charge.

The good ghost is not restricted from entering physical states by the Gupta condition but is nonetheless never observed directly as a free excitation. The reason for this becomes clear when the expectation value of the vector potential is considered in a state of the form $|\alpha) = |0) + |g)$; then it is found that the effect of the good ghost is merely to add a 4-divergence to the expectation value (ignoring a delta function at the surface). Thus the addition of a good ghost is equivalent to a gauge change of the classical theory, and hence is as undetectable as the gauge change itself. (See also Itzykson and Zuber (1980).)

3. The transformation and the image potential

The Lorentz gauge is manifestly Lorentz invariant but the Coulomb gauge is not, hence in transforming from one to the other it is necessary to fix the Lorentz frame. The required frame is that in which the transverse and longitudinal modes have no time-like component and the scalar mode no space-like components. In this special frame the generalised transverse, longitudinal and scalar modes become equivalent to the corresponding classical modes, and the polarisation vectors of the ghost states become

$$\varepsilon^{\mu g}(k) = k^\mu/\omega\sqrt{2} \qquad \varepsilon^{\mu b}(k) = \hat{k}^\mu/\omega\sqrt{2} \tag{26}$$

where, as defined previously, $\hat{k}^\mu = (\omega, \boldsymbol{k}_\parallel, -k_\perp)$.

A unitary transformation U is employed, taking the usual form in which an arbitrary operator Q and a state vector $|\alpha)$ are transformed according to

$$|\alpha)' = U|\alpha) \qquad Q' = UQU^{-1}. \tag{27}$$

A unitary transformation is a non-singular transformation that leaves the metric intact: $(U\phi|U\psi) = (\phi|\psi)$. This definition is as valid when used with an indefinite metric as with any other non-degenerate metric. Unitarity with respect to an indefinite metric is merely a straightforward generalisation of ordinary unitarity, it is in no sense a false unitarity, even though the term pseudo-unitarity is often applied to it.

A unitary operator can be put in the form $U = e^{-i\sigma}$ where σ is a self-adjoint operator. In the present case a suitable form for σ is found to be

$$\sigma = q\int_{\frac{1}{2}} d^3\boldsymbol{k}\, D(k)a^g(k)\exp(i\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_\parallel)\sin(k_\perp r_\perp) + \text{adjoint} \tag{28}$$

in which $D(k) = M(k)/\omega\sqrt{2}$. It is evident in a general way that U is a clothing operator, since it links the particle position operator r^μ with the ghost-state operators. An equivalent operator constructed for the relativistic theory of the full-space is given by Durr and Rudolph (1969).

The transformation is applied to the various component parts of the Hamiltonian in turn, using the Schrödinger picture. It is first noticed that $a^{(1)}$, $a^{(2)}$ and a^{g} commute with σ and thus transform into themselves. On the other hand, using the commutation relations (20) and performing the integral, gives

$$[\sigma, a^{b}(k)] = -qD(k)\exp(\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{r}_{\parallel})\sin(k_{\perp}r_{\perp}). \tag{29}$$

This commutator itself commutes with σ and hence, using a well known formula,

$$\begin{aligned} a^{b\prime}(k) &= \mathrm{e}^{-\mathrm{i}\sigma}a^{b}(k)\,\mathrm{e}^{\mathrm{i}\sigma} \\ &= a^{b}(k) + \mathrm{i}qD(k)\exp(-\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{r}_{\parallel})\sin(k_{\perp}r_{\perp}). \end{aligned} \tag{30}$$

From equation (25) it is then immediately apparent that the Gupta condition in the transformed theory is simply

$$a^{b}|\text{physical})' = 0 \tag{31}$$

valid now whether or not charged particles are present.

The particle position vector r^{μ} commutes with σ and thus transforms into itself. The momentum operator, however, does not:

$$\boldsymbol{p}' = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\sigma}\boldsymbol{\nabla}_{r}(\mathrm{e}^{\mathrm{i}\sigma}) = \boldsymbol{p} + \boldsymbol{\nabla}_{r}\sigma \tag{32}$$

using the fact that $\boldsymbol{\nabla}_{r}\sigma$ commutes with σ. Differentiating σ and expressing the result in the special frame of equations (26) gives for $r_{\perp} > 0$

$$\boldsymbol{p}' = \boldsymbol{p} + q\boldsymbol{A}^{g}(r) \tag{33}$$

where $\boldsymbol{A}^{g}$ is the 3-vector part of the good-ghost vector potential of equation (17).

Equations (30) and (33) show how intimate is the relationship in the transformed theory between the ghost states and electric charge.

The number operators are easily transformed by simple substitution of equation (30) into equation (21):

$$N^{g\prime} = N^{g} - \mathrm{i}qD(k)a^{g}(k)\exp(\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{r}_{\parallel})\sin(k_{\perp}r_{\perp}) \tag{34a}$$

$$N^{b\prime} = N^{b} + \mathrm{i}qD(k)a^{g+}(k)\exp(-\mathrm{i}\boldsymbol{k}_{\parallel}\cdot\boldsymbol{r}_{\parallel})\sin(k_{\perp}\boldsymbol{r}_{\perp}) \tag{34b}$$

whilst the number operators of the transverse states transform into themselves.

The use of equations (34) in equation (1*b*) gives H'_{field}, and then employing equations (26) leads to

$$H'_{\text{field}} = H_{\text{field}} - qA^{0g}(r). \tag{35}$$

It remains to transform the vector potential itself; however, only the part $A^{\mu b}$, equation (18), needs much consideration as the transverse and good-ghost parts commute with σ and hence transform into themselves. The space-like components of $A^{\mu b}$ also transform into themselves as is seen on using equations (30) and (26), and considering symmetry about the normal to the mirror surface.

The time component behaves differently. It occurs in the Hamiltonian only in the term H_{ϕ} of equation (1*d*). This is the term that leads to the Coulomb-energy term under the transformation; however, to see this most clearly it is necessary to generalise to the case where more than one particle is in interaction with the field. Then H_{ϕ} becomes

$$H_{\phi} = \sum_{i} q_{i}A^{0}(r_{i}) \tag{36}$$

where r_i^μ is the position vector of particle i with charge q_i. Correspondingly, equation (30) becomes

$$a^{b\prime}(k)=a^b(k)+\mathrm{i}\sum_j q_j D(k)\exp(-\mathrm{i}\boldsymbol{k}_\parallel\cdot\boldsymbol{r}_{\parallel j})\sin(k_\perp r_{\perp j}). \tag{37}$$

Putting these equations together and transforming to the special frame of equations (26) gives

$$H'_\Phi=H_\Phi+\frac{1}{8\pi^2}\sum_{ij}q_iq_j[I(\boldsymbol{r}_i,\boldsymbol{r}_j)+\text{adjoint}] \tag{38}$$

where

$$I(\boldsymbol{r}_i,\boldsymbol{r}_j)=\int_{\frac{1}{2}}\mathrm{d}^3\boldsymbol{k}\frac{1}{\boldsymbol{k}^2}\exp[\mathrm{i}\boldsymbol{k}_\parallel\cdot(\boldsymbol{r}_{\parallel i}-\boldsymbol{r}_{\parallel j})]\sin(k_\perp r_{\perp i})\sin(k_\perp r_{\perp j}). \tag{39}$$

Expressing the sine functions in terms of exponentials, changing the limits of integration and rearranging, leads to

$$I(\boldsymbol{r}_i,\boldsymbol{r}_j)=\tfrac{1}{4}\int_{\text{full-space}}\mathrm{d}^3\boldsymbol{k}\frac{1}{\boldsymbol{k}^2}\{\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)]-\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i-\hat{\boldsymbol{r}}_j)]\} \tag{40}$$

where $\hat{\boldsymbol{r}}=(t,\boldsymbol{r}_\parallel,-r_\perp)$, as defined previously. Then changing to polar coordinates, simplifying and making use of the tabulated integrals of Abramowitz and Stegun (1964), eventually gives

$$H'_\Phi=H_\Phi+H_{\text{Cb}}+H_{\text{image}} \tag{41a}$$

where

$$H_{\text{Cb}}=\tfrac{1}{2}\sum_{ij}\frac{q_iq_j}{4\pi[(\boldsymbol{r}_{\parallel i}-\boldsymbol{r}_{\parallel j})^2+(r_{\perp i}-r_{\perp j})^2]^{1/2}} \tag{41b}$$

and

$$H_{\text{image}}=\tfrac{1}{2}\sum_{ij}\frac{-q_iq_j}{4\pi[(\boldsymbol{r}_{\parallel i}-\boldsymbol{r}_{\parallel j})^2+(r_{\perp i}+r_{\perp j})^2]^{1/2}}. \tag{41c}$$

The term H_{Cb} is the ordinary Coulomb energy of a system of charges expressed in Heaviside units. The factor $\frac{1}{2}$ relates to the double appearance of each pair of charges in the sum over pairs. The other term, H_{image}, is identical apart from the changes in sign of two quantities, q_j and $r_{\perp j}$. This term is the classical image charge term, representing the energy of interaction of each charge with a set of image charges.

The image energy of a single charge is simply given by one of the terms of H_{image} in which $i=j$:

$$E_{\text{image}}=-\left(\frac{1}{4\pi}\right)\frac{q^2}{4r_\perp}. \tag{42}$$

This result is identical to the famous classical result.

The complete Hamiltonian becomes

$$\begin{aligned}H'=H^{\text{Tr}}_{\text{field}}+H^{\text{g}}_{\text{field}}+H^{\text{b}}_{\text{field}}+\frac{1}{2m}[\boldsymbol{p}-q(\boldsymbol{A}^{\text{Tr}}(r)+\boldsymbol{A}^{\text{b}}(r))]^2\\+q(A^{0\text{Tr}}(r)+A^{0\text{b}}(r))+H_{\text{Cb}}+H_{\text{image}}\end{aligned} \tag{43}$$

where H_{field} has been split into its transverse and ghost components.

4. Ghost states in the Coulomb gauge

The transformed Hamiltonian of equation (43) should be compared with the Coulomb-gauge Hamiltonian as it normally appears

$$H_{\text{Coulomb}} = H^{\text{Tr}}_{\text{field}} + \frac{1}{2m}(\boldsymbol{p} - \boldsymbol{A}^{\text{Tr}}(\boldsymbol{r}))^2 + qA^{0\text{Tr}}(\boldsymbol{r}) + H_{\text{Cb}} + H_{\text{image}}. \tag{44}$$

The two Hamiltonians are very nearly the same, but not quite, since H' contains terms not appearing in H_{Coulomb}. These terms are first the free-field terms $H^{\text{g}}_{\text{field}}$ and $H^{\text{b}}_{\text{field}}$ which might have been expected not to vanish in the transformation. Second, however, there are the ghost terms $\boldsymbol{A}^{\text{b}}$ and $A^{0\text{b}}$ appearing in H' which would seem to prevent the ghost states decoupling. However, a Hamiltonian without these terms does not seem to be derivable by unitary transformation without at the same time losing the Coulomb terms. It seems that the transformed theory is a version of the Coulomb gauge in which two virtual photons exist. Neither ghost state is directly detectable by physical measurement in the Coulomb gauge any more than it would be in the Lorentz gauge, and for the same reasons; in fact the Gupta condition in the transformed theory is stronger than that in the Lorentz gauge as it prohibits entry of the bad ghost to physical states whether or not electric charge is present. The ghosts do have an effect on the theory of the Coulomb gauge for they clearly add a component to the zero-point energy. However the presence of ghost states does not alter any other energy level of the system away from what would be expected in the usual formulation of the Coulomb gauge and it is the purpose of the next section to prove this. It is not clear at present whether or not the ghosts are in other respects completely decoupled, although there is no reason to assume that they are not; a proof might be constructed along the lines employed in § 4.1 generalised to the expectation values of operators other than the Hamiltonian.

4.1. Proof that the ghost states have no effect on energy levels

That the energy levels of the transformed system are identical to those of the Coulomb gauge can be shown using an expansion of perturbation theory type. The transformed Hamiltonian H' can be put in the form

$$H' = H_0 + V \tag{45a}$$

where

$$H_0 = H_{\text{Coulomb}} + H^{\text{g}}_{\text{field}} + H^{\text{b}}_{\text{field}} \tag{45b}$$

and

$$V = \frac{1}{2m}\{q^2\boldsymbol{A}^{\text{b}}(\boldsymbol{r})\cdot\boldsymbol{A}^{\text{b}}(\boldsymbol{r}) - q[\boldsymbol{p}\cdot\boldsymbol{A}^{\text{b}}(\boldsymbol{r}) + \boldsymbol{A}^{\text{b}}(\boldsymbol{r})\cdot\boldsymbol{p}] + 2q^2\boldsymbol{A}^{\text{Tr}}(\boldsymbol{r})\cdot\boldsymbol{A}^{\text{b}}(\boldsymbol{r})\} + qA^{0\text{b}}(\boldsymbol{r}) \tag{45c}$$

with H_0 identical to the standard Coulomb-gauge Hamiltonian apart from the addition of the free-field ghost terms. If now $|Cb)$ represents the eigenstates of the standard Coulomb-gauge Hamiltonian then the eigenstates of H_0 must take the form $|i_0) = |n^{\text{g}}, n^{\text{b}})|Cb)$, in which $|Cb)$ represents a coupled system of electron wavefunction and transverse states but is not coupled to the ghost states.

Proceeding as usual in time-independent perturbation theory, it is assumed that the true eigenstates $|i)$ and eigenvalues E_i of H' can be expanded as power series in a perturbation parameter λ such that

$$|i) = |i_0) + \lambda|i_1) + \lambda^2|i_2) + \ldots \qquad E_i = E_{i_0} + \lambda E_{i_1} + \lambda^2 E_{i_2} + \ldots$$

where

$$[H_0 + \lambda V]|i) = E_i|i) \qquad H_0|i_0) = E_{i_0}|i_0).$$

Adapting the normal methods of perturbation theory to the needs of the indefinite metric gives to first order

$$|i_1) = \sum_{j \neq i} \frac{(\hat{j}_0|V|i_0)}{E_{i_0} - E_{j_0}}|j_0) \tag{46a}$$

$$E_{i_1} = (\hat{i}_0|V|i_0) \tag{46b}$$

where states such as $|\hat{i})$ and $|i)$ are related through equations (24). Then to nth order where $n \geqslant 2$

$$|i_n) = \sum_{j \neq i} \frac{(\hat{j}_0|E_{i_1} - V|i_{n-1}) + \sum_{r=2}^{n} E_{i_r}(\hat{j}_0|i_{n-r})}{E_{j_0} - E_{i_0}}\Bigg|j_0\Bigg) \tag{47a}$$

$$E_{i_n} = (\hat{i}_0|V|i_{n-1}) \tag{47b}$$

where the sum over r in equation (47a) always has at least one term, even if $n = 2$.

The perturbation V contains the ghost-state operators in the combinations: a^{b}, Cr^{b}, $a^{\mathrm{b}}a^{\mathrm{b}}$, $Cr^{\mathrm{g}}Cr^{\mathrm{g}}$ and $Cr^{\mathrm{g}}a^{\mathrm{b}}$. Inserting these combinations of operators into equation (46a) it is found that the only action of the perturbation to first order when $|i_0)$ represents a physical state ($n^{\mathrm{b}} = 0$) is to add linear combinations of terms in which good ghosts are present but bad ghosts are not. Thus the perturbation does not transform a physical state out of the set of physical states. From equation (46b) it is determined in a similar manner that the energy eigenvalue of a physical state, and hence the expectation value, remains unaltered to first order under the action of the perturbation.

The process can be repeated to arbitrary order by induction. Firstly it is assumed that for some order n, where $n \geqslant 2$, that $|i_{n-1})$ is a physical state, and that $E_{i_r} = 0$ for all $r < n$. Then using equation (47b) it is found in a similar manner to the above that $E_{i_n} = 0$, and hence the second term on the right-hand side of equation (47a) vanishes. From the resulting equation it is found that if $|i_{n-1})$ is a physical state then $|i_n)$ is also such a state. Thus if the assumptions hold to order $n-1$ they also hold to order n, and since it has been shown that they hold to first order the usual process establishes that they hold to all orders. Thus the expectation values of the transformed Hamiltonian in physical states are not altered by interactions with the ghost states, and hence are identical to those that would be obtained from the Coulomb-gauge Hamiltonian in its more usual form. This constitutes the proof required.

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